

LATTICE POINTS IN ROTATED CONVEX DOMAINS

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ABSTRACT. If $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain with a smooth boundary of finite type, we prove that for almost every rotation $\theta \in SO(d)$ the remainder of the lattice point problem, $P_{\theta\mathcal{B}}(t)$, is of order $O_\theta(t^{d-2+2/(d+1)-\zeta_d})$ with a positive number ζ_d . Furthermore we extend the estimate of the above type, in the planar case, to general compact convex domains.

1. INTRODUCTION

Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) be a compact convex domain, which contains the origin in its interior and has a smooth boundary $\partial\mathcal{B}$. The number of lattice points \mathbb{Z}^d in the dilated domain $t\mathcal{B}$ is approximately $|t\mathcal{B}|$ (*i.e.* the volume (area if $d = 2$) of $t\mathcal{B}$) and the lattice point problem is to study the remainder, $P_{\mathcal{B}}(t)$, in the equation

$$P_{\mathcal{B}}(t) = \#(t\mathcal{B} \cap \mathbb{Z}^d) - |t\mathcal{B}| \quad \text{for } t \geq 1.$$

A trivial estimate gives $P_{\mathcal{B}}(t) = O(t^{d-1})$.

If $\partial\mathcal{B}$ has everywhere positive (Gaussian) curvature, a standard estimate is

$$P_{\mathcal{B}}(t) = O(t^{d-2+2/(d+1)}),$$

which can be readily obtained by a combination of the Poisson summation formula and (nowadays standard) oscillatory integral estimates (see Hlawka [6]). Over the years this result has been improved by many authors and the best bounds up-to-date are due to Huxley [8] in the planar case and Guo [4] in the higher dimensional case. For a survey on historical results the reader is referred to Ivić, Krätzel, Kühleitner, and Nowak [11].

While the above case is relatively well understood, the general case when the (Gaussian) curvature is allowed to vanish is not.

Let us first consider the $d \geq 3$ case of vanishing curvature. Partial results indicate that the remainder may become much larger. For example

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Randol [24] considered the super spheres

$$\mathcal{B} = \{(x_1, \dots, x_d) \in \mathbb{R}^d : |x_1|^\omega + |x_2|^\omega + \dots + |x_d|^\omega \leq 1\}$$

for even integer $\omega \geq 3$, and proved that

$$P_{\mathcal{B}}(t) = \begin{cases} O(t^{d-2+2/(d+1)}) & \text{for } \omega \leq d+1, \\ O(t^{(d-1)(1-1/\omega)}) & \text{for } \omega > d+1, \end{cases}$$

and this estimate is the best possible when $\omega > d+1$. Krätzel [12] extended this result to odd $\omega \geq 3$ and gave an asymptotic formula

$$(1.1) \quad P_{\mathcal{B}}(t) = H(t)t^{(d-1)(1-1/\omega)} + O(t^\Theta)$$

with an explicit $\Theta < (d-1)(1-1/\omega)$ and $H(t)$ continuous and periodic (see Krätzel [13] for more details). We observe that the remainder $P_{\mathcal{B}}(t)$ becomes extremely large as $\omega \rightarrow \infty$.

This observation is supported by the study of more examples, and special attention is paid to specific convex domains in \mathbb{R}^3 . See Krätzel [16] and Krätzel and Nowak [17, 18], in which they proved, among other results, asymptotic formulas of $P_{\mathcal{B}}(t)$ with explicit representations of the main terms given.

For general domains with boundary points of Gaussian curvature zero, our knowledge is still very poor. Partial results in \mathbb{R}^3 are available in Krätzel [14, 15], Peter [22], Popov [23], and Nowak [21] (with the latter two papers focusing on bodies of rotation). Under a variety of assumptions, they provide O -estimates (or asymptotic formulas) of $P_{\mathcal{B}}(t)$, and evaluate the contributions (to $P_{\mathcal{B}}(t)$) of different types of boundary points of Gaussian curvature zero. Their results show that the size of $P_{\mathcal{B}}(t)$ depends on certain properties of the boundary points of Gaussian curvature zero and whether the slope of the normal at such a point is rational or irrational. In particular $P_{\mathcal{B}}(t)$ may become extremely large and a substantial contribution to it is due to the neighborhoods of those boundary points of Gaussian curvature zero at which the normal has a rational direction.

However after a rotation of the domain there may be no such points, hence we can expect a better estimate. For example Iosevich, Sawyer, and Seeger [10] proved, for convex domains of finite type ⁽¹⁾, that there is $r > 2$ so that

$$(1.2) \quad P_{\mathcal{B}_\theta}(t) = O_\theta(t^{d-2+2/(d+1)} \log^{1/r}(2+t)) \quad \text{for a.e. } \theta \in SO(d),$$

where $\mathcal{B}_\theta = \theta\mathcal{B}$ denotes the rotated domain $\{\theta x : x \in \mathcal{B}\}$. Results of type (1.2) with the same exponent $d-2+2/(d+1)$ can be found in Randol [25] for convex domains with an analytic boundary, and in Colin de Verdière [3] for general (not necessarily convex) domains if $d \leq 7$.

It is then natural to ask whether one can prove a result of type (1.2) with an exponent $d-2+2/(d+1)-c$ for some positive c . We made a progress in

¹Check §2 for its definition.

this direction and proved the following theorem with a $c > 0$ depending only on the dimension d .

Theorem 1.1. *Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 3$) be a compact convex domain containing the origin in its interior. If the boundary is a smooth hypersurface of finite type then*

$$P_{\mathcal{B}_\theta}(t) = O_\theta(t^{d-2+2/(d+1)-\zeta_d}) \quad \text{for a.e. } \theta \in SO(d),$$

where $\zeta_d > 0$ is defined as

$$(1.3) \quad \zeta_d = \begin{cases} \frac{2(d-2)(d-1)d}{(d+1)(6d^5+118d^4+109d^3-210d^2-119d+82)} & \text{for } 3 \leq d \leq 4, \\ \frac{(d-3)(d-1)d}{2d^6+49d^5+123d^4-9d^3-167d^2-52d+30} & \text{for } d \geq 5. \end{cases}$$

This result is an easy consequence of the following theorem.

Theorem 1.2. *Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 3$) be a compact convex domain containing the origin in its interior. If the boundary is a smooth hypersurface of finite type ω then*

$$\sup_{t \geq 2} |P_{\mathcal{B}_\theta}(t)| / (t^{d-2+2/(d+1)-\zeta_d-\sigma(d,\omega)} \log^b(t)) \in L^1(SO(d)),$$

where $b > 1$, ζ_d is given by (1.3), and $\sigma(d, \omega) > 0$ is defined as

$$(1.4) \quad \sigma(d, \omega) = \begin{cases} \frac{4d(6d^5+100d^4-230d^3-193d^2+496d-172)}{(6d^5+118d^4+109d^3-210d^2-119d+82) \cdot \square} & \text{for } 3 \leq d \leq 4, \\ \frac{2d(2d^5+39d^4-105d^3-205d^2+377d-96)}{(2d^5+47d^4+76d^3-85d^2-82d+30) \cdot \triangle} & \text{for } d \geq 5, \end{cases}$$

with

$$(1.5) \quad \begin{aligned} \square &= 6(\omega-2)d^5 + 118(\omega-2)d^4 + 109(\omega-2)d^3 \\ &\quad - 6(35\omega-71)d^2 + (246-119\omega)d + (82\omega-156) \end{aligned}$$

and

$$(1.6) \quad \begin{aligned} \triangle &= 2(\omega-2)d^5 + 47(\omega-2)d^4 + 76(\omega-2)d^3 \\ &\quad + (172-85\omega)d^2 + (166-82\omega)d + (30\omega-56). \end{aligned}$$

The proof of Theorem 1.2 relies on the following analysis result (implied by Svensson's [28, Theorem 4.1]): if $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 3$) is a compact convex domain and its boundary is a smooth hypersurface of finite type ω ⁽²⁾ then

$$(1.7) \quad \Phi \in L^p(S^{d-1}) \quad \text{for } 2 < p < 2 + 2/(d-1)(\omega-2),$$

where

$$(1.8) \quad \Phi(\xi) = \sup_{r>0} r^{(d+1)/2} |\widehat{\chi}_{\mathcal{B}}(r\xi)|, \quad \xi \in S^{d-1}.$$

²The restriction on the size of ω given by Svensson's [28, Theorem 4.1] can be removed under current assumptions.

For a general convex domain \mathcal{B} with a smooth boundary, (1.7) is not necessarily true, however, we always have (due to Varchenko's [29, Theorem 8]) that

$$r^{(d+1)/2} |\widehat{\chi}_{\mathcal{B}}(r\xi)| \in L^2(S^{d-1}).$$

By using this result we can readily modify the proof of Theorem 1.2 and prove the following theorem, which improves similar results contained in Randol [25, p. 285] and Varchenko's [29, Theorem 7] in terms of the estimate.

Theorem 1.3. *Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 3$) be a compact convex domain containing the origin in its interior. If the boundary is a smooth hypersurface then*

$$|P_{\mathcal{B}_\theta}(t)|/t^{d-2+2/(d+1)-\zeta_d} \in L^1(SO(d)),$$

where ζ_d is given by (1.3).

Let us now consider the $d = 2$ case of vanishing curvature, in which we have a better understanding than in higher dimensions. We refer the interested readers to Ivić, Krätzel, Kühleitner, and Nowak [11] and Guo [5] for an introduction to related results.

For general convex planar domains we know $\Phi \in L^{2,\infty}(S^1)$ (Brandolini, Colzani, Iosevich, Podkorytov, and Travaglini's [2, Theorem 0.3]). By using this result and the same method used in the proofs of Theorem 1.2 and 1.3, we are able to extend our previous result for convex planar domains of finite type in Guo's [5, Theorem 1.1] to the following result for convex planar domains with no curvature assumption on the boundary (with even a better estimate, due to an improved estimate of certain nonvanishing determinants given in Lemma 3.5 below).

Theorem 1.4. *If \mathcal{B} is a compact convex planar domain with a smooth boundary containing the origin in its interior, then*

$$\sup_{t \geq 2} |P_{\mathcal{B}_\theta}(t)|/(t^{2/3-\zeta_2} \log^b(t)) \in L^1(SO(2)),$$

where $b > 1$ and $\zeta_2 = 1/2859$. In particular,

$$P_{\mathcal{B}_\theta}(t) = O_\theta(t^{2/3-\zeta_2} \log^b(t)) \quad \text{for a.e. } \theta \in SO(2).$$

This theorem improves Iosevich's [9, Theorem 0.2] and also Brandolini, Colzani, Iosevich, Podkorytov, and Travaglini's [2, Theorem 0.1] (in terms of the estimate). If we assume that the boundary is of finite type then we have the following better estimate (again due to the improved result given in Lemma 3.5 below), which improves Guo's [5, Theorem 1.2].

Theorem 1.5. *If \mathcal{B} is a compact convex planar domain with a smooth boundary of finite type ω containing the origin in its interior, then*

$$\sup_{t \geq 2} |P_{\mathcal{B}_\theta}(t)|/(t^{2/3-\zeta_2-\sigma(2,\omega)} \log^b(t)) \in L^1(SO(2)),$$

where $b > 1$, $\zeta_2 = 1/2859$, and

$$\sigma(2, \omega) = \frac{616}{953(953\omega - 1848)}.$$

In particular,

$$P_{\mathcal{B}_\theta}(t) = O_\theta(t^{2/3-\zeta_2}) \quad \text{for a.e. } \theta \in SO(2).$$

Remark 1.6. Our main idea originates from Iosevich, Sawyer, and Seeger [10, p. 168-169] and Müller [20]. Our main tools used in this paper are from the oscillatory integral theory and the classical Van der Corput's method of exponential sums (namely, the A - and B -processes). To prove our estimate of exponential sums (see Proposition B.1 below) we use an $A^q B$ -process. If we use more A - and B -processes we may achieve further improvement at the cost of more technical difficulties.

Notations: We use the usual Euclidean norm $|x|$ for a point $x \in \mathbb{R}^d$. $B(x, r)$ represents the Euclidean ball centered at x with radius r , and its dimension will be clear from the context. The norm of a matrix $A \in \mathbb{R}^{d \times d}$ is given by $\|A\| = \sup_{|x|=1} |Ax|$. We set $e(f(x)) = \exp(-2\pi i f(x))$, $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$, and $\mathbb{R}_*^d = \mathbb{R}^d \setminus \{0\}$. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is given by $\hat{f}(\xi) = \int f(x) e(\langle x, \xi \rangle) dx$.

We fix χ_0 to be a smooth cut-off function whose value is 1 on $B(0, 1/2)$ and 0 on the complement of $B(0, 1)$. For a set $E \subset \mathbb{R}^d$ and a positive number a , we define $E_{(a)}$ to be the larger set

$$E_{(a)} = \{x \in \mathbb{R}^d : \text{dist}(E, x) < a\}.$$

We use the differential operators

$$D_x^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}} \quad (\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d, |\nu| = \sum_{i=1}^d \nu_i)$$

and the gradient operator ∇_x . We often omit the subscript if no ambiguity occurs.

Structure of the paper: We first establish some preliminaries in §2-4 mainly for compact convex domains with no curvature assumption on the smooth boundary. We then give a proof of Theorem 1.2 in §5, in which the problem is reduced to the estimate of two sums (Sum I and II). The estimate of Sum I that we give essentially works for general compact convex domains, while the curvature condition on the boundary is used in the estimate of Sum II. Since it is easy to modify the proof of Theorem 1.2 to prove the other theorems, we only provide brief proofs of Theorem 1.3 in §5 and Theorem 1.4 and 1.5 in §6. At last we collect some standard analysis results in Appendix A and prove an estimate of exponential sums in Appendix B for interested readers.

2. SOME GEOMETRIC FACTS

Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$). For a point $x \in \partial\mathcal{B}$, let $K(x)$ be the (Gaussian) curvature of $\partial\mathcal{B}$ at x . Define

$$(\partial\mathcal{B})_+ = \{x \in \partial\mathcal{B} : K(x) > 0\} \quad \text{and} \quad (\partial\mathcal{B})_0 = \{x \in \partial\mathcal{B} : K(x) = 0\},$$

thus

$$\partial\mathcal{B} = (\partial\mathcal{B})_+ \bigsqcup (\partial\mathcal{B})_0.$$

The Gauss map of $\partial\mathcal{B}$, denoted by \vec{n} , maps each boundary point $x \in \partial\mathcal{B}$ to a unit exterior normal $\vec{n}(x) \in S^{d-1}$. Then

$$S^{d-1} = \vec{n}((\partial\mathcal{B})_+) \bigsqcup \vec{n}((\partial\mathcal{B})_0).$$

Note that the restriction of \vec{n} to $(\partial\mathcal{B})_+$, namely

$$\vec{n}|_{(\partial\mathcal{B})_+} : (\partial\mathcal{B})_+ \rightarrow \vec{n}((\partial\mathcal{B})_+) \subset S^{d-1},$$

is bijective. For $\xi \neq 0$ with $\xi/|\xi| \in \vec{n}((\partial\mathcal{B})_+)$ let $x(\xi) := \vec{n}^{-1}(\xi/|\xi|)$ be the unique point on $\partial\mathcal{B}$ where the unit exterior normal is $\xi/|\xi|$. Hence $K_\xi = K(x(\xi))$ is well defined for such points ξ .

For $\xi \neq 0$ with $\xi/|\xi| \in \vec{n}((\partial\mathcal{B}_\theta)_+)$ let $x^\theta(\xi) = \theta x(\theta^t \xi)$ and $K_\xi^\theta = K_{\theta^t \xi}$. Then $x^\theta(\xi)$ is the unique point on $\partial\mathcal{B}_\theta$ where the exterior normal is ξ and K_ξ^θ is the curvature of $\partial\mathcal{B}_\theta$ at $x^\theta(\xi)$.

Lemma 2.1. *Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$). Then there exists a constant $c_1 > 0$ (depending only on \mathcal{B}) such that, for any $\xi \in \vec{n}((\partial\mathcal{B})_+)$, if $\eta \in B(\xi, c_1(K_\xi)^2) \subset \mathbb{R}^d$ then $\eta/|\eta| \in \vec{n}((\partial\mathcal{B})_+)$ and*

$$K_\xi/2 \leq K_\eta \leq 3K_\xi/2.$$

Proof. For any $\xi \in \vec{n}((\partial\mathcal{B})_+)$ it follows from the mean value theorem that there exists a constant c (depending only on \mathcal{B}) such that

$$K_\xi/2 \leq K(y) \leq 3K_\xi/2 \quad \text{if } y \in B(x(\xi), cK_\xi) \cap \partial\mathcal{B}.$$

It is a consequence of Lemma A.1 that the Gauss map is bijective from a subset of $B(x(\xi), cK_\xi) \cap \partial\mathcal{B}$ onto a subset of S^{d-1} containing $B(\xi, c'(K_\xi)^2) \cap S^{d-1}$ where the constant c' depends only on \mathcal{B} . Then the lemma follows easily. \square

Lemma 2.2. *Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$). Then*

$$|\vec{n}(\{x \in \partial\mathcal{B} : K(x) < \delta\})| \leq C_{\mathcal{B}} \delta |\{x \in \partial\mathcal{B} : 0 < K(x) < \delta\}|$$

where the absolute value denotes the induced Lebesgue measure on surfaces (curves if $d = 2$).

Proof. Note that

$$\{x \in \partial\mathcal{B} : K(x) < \delta\} = \{x \in \partial\mathcal{B} : 0 < K(x) < \delta\} \bigcup (\partial\mathcal{B})_0.$$

We first have

$$|\vec{n}((\partial\mathcal{B})_0)| = 0$$

due to Sard's theorem (see Lang [19, p. 286]). Hence it suffices to prove

$$|\vec{n}(\{x \in \partial\mathcal{B} : 0 < K(x) < \delta\})| \leq C_{\mathcal{B}}\delta|\{x \in \partial\mathcal{B} : 0 < K(x) < \delta\}|.$$

By using a standard technique found in the proof of certain covering lemma of Vitali type (see Stein [27]), we reduce the above estimate to

$$|\vec{n}(B)| \leq C_{\mathcal{B}}\delta|B|,$$

where $B \subset \{x \in \partial\mathcal{B} : 0 < K(x) < \delta\}$ is a ball in $\partial\mathcal{B}$. However this last estimate follows from the equality $d\sigma = K(x)dA$ where dA is the volume element of $\partial\mathcal{B}$ at the point $x \in \partial\mathcal{B}$ and $d\sigma$ the volume element of S^{d-1} at the point $\vec{n}(x) \in S^{d-1}$ (see [1, p. 47]; this equality can also be verified by using local coordinate charts). This finishes the proof. \square

We say that the boundary $\partial\mathcal{B}$ is of finite type if at every point $x \in \partial\mathcal{B}$, every one dimensional tangent line to $\partial\mathcal{B}$ at x makes finite order of contact with $\partial\mathcal{B}$. If $\partial\mathcal{B}$ is of finite type, the maximum order of contact over all $x \in \partial\mathcal{B}$ and all tangent lines to $x \in \partial\mathcal{B}$ is called the type of $\partial\mathcal{B}$.

We will always assume below that the type is ≥ 3 since if the type is two then we recover the case of nonvanishing (Gaussian) curvature.

Lemma 2.3. *Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$) of finite type ω . Then*

$$|\{x \in \partial\mathcal{B} : K(x) < \delta\}| \leq C_{\mathcal{B}}\delta^{1/(d-1)(\omega-2)}.$$

Proof. By using a compactness argument and local coordinates we may only regard K as a function of x' in a neighborhood $B(0, C_0)$ of 0 in \mathbb{R}^{d-1} for some constant C_0 . We may assume that $K, \partial K/\partial x_1, \dots, \partial^h K/\partial x_1^h$ (with $h = (d-1)(\omega-2)$) do not vanish simultaneously (see Svensson [28, p. 19]). We then apply Svensson's [28, Lemma 3.3] to K in x_1 -direction, which yields

$$|\{x_1 : |x_1| \leq C_0, K(x') < \delta\}| \leq C_{\mathcal{B}}\delta^{1/h},$$

and the trivial estimate in x_2, \dots, x_{d-1} -directions. Thus the desired estimate follows. \square

3. NONVANISHING $d \times d$ DETERMINANTS

In this section we always assume that $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$).

The support function of \mathcal{B} is given by $H(\xi) = \sup_{y \in \mathcal{B}} \langle \xi, y \rangle$ for any nonzero $\xi \in \mathbb{R}^d$. In particular $H(\xi) = \langle \xi, x(\xi) \rangle$ for any nonzero ξ with $\xi/|\xi| \in$

$\vec{n}((\partial\mathcal{B})_+)$. It is positively homogeneous of degree one, i.e. $H(\lambda\xi) = \lambda H(\xi)$ if $\lambda > 0$.

The next two lemmas can be easily proved by using local coordinates, hence we omit the proof.

Lemma 3.1. *H is smooth at every $\xi \in \vec{n}((\partial\mathcal{B})_+)$ and satisfies*

$$D^\nu H(\xi) \lesssim 1 \quad \text{for } 0 \leq |\nu| \leq 1$$

and

$$D^\nu H(\xi) \lesssim (K_\xi)^{3-2|\nu|} \quad \text{for } |\nu| \geq 2,$$

where implicit constants may depend only on $|\nu|$ and \mathcal{B} .

Remark 3.2. For $\theta \in SO(d)$, we will denote the support function of \mathcal{B}_θ by $H_\theta(\xi) = \sup_{y \in \mathcal{B}_\theta} \langle \xi, y \rangle$. Since $H_\theta(\xi) = H(\theta^t \xi)$, we can easily get bounds for H_θ in the same form as in the above lemma (with $\vec{n}((\partial\mathcal{B})_+)$ and K_ξ replaced by $\theta \vec{n}((\partial\mathcal{B})_+)$ and K_ξ^θ respectively).

Lemma 3.3. *Assume that $\xi \neq 0$ with $\xi/|\xi| \in \vec{n}((\partial\mathcal{B})_+)$. If $d \geq 3$ the eigenvalues of the matrix $\nabla_{\xi\xi}^2 H(\xi)$ are 0, $(|\xi|\kappa_1)^{-1}$, \dots , $(|\xi|\kappa_{d-1})^{-1}$, where $\{\kappa_j\}_{j=1}^{d-1}$ are the principle curvatures of $\partial\mathcal{B}$ at $x(\xi)$; if $d = 2$ the eigenvalues are 0 and $(|\xi|K_\xi)^{-1}$.*

Given d vectors $v_1, \dots, v_d \in \mathbb{R}^d$, by writing $V = (v_1, \dots, v_d)$ we mean V is the matrix in $\mathbb{R}^{d \times d}$ with column vectors v_1, \dots, v_d . If $y \neq 0$ we define $F_\theta(u_1, \dots, u_d) = H_\theta(y + \sum_{l=1}^d u_l v_l)$, $u_l \in \mathbb{R}$ ($l = 1, \dots, d$). For $q \in \mathbb{N}$ let

$$h_q^\theta(y, v_1, \dots, v_d) = \det(g_{i,j}^\theta(y, v_1, \dots, v_d))_{1 \leq i,j \leq d},$$

where

$$g_{i,j}^\theta(y, v_1, \dots, v_d) = \frac{\partial^{q+2} F_\theta}{\partial u_1 \partial u_i \partial u_j \partial u_d^{q-1}}(0).$$

The following lemma is a higher dimensional analogue of Guo's [5, Lemma 3.4], which enables us to apply the method of stationary phase later in the estimate of certain exponential sums. The result is for $\xi \in \vec{n}((\partial\mathcal{B}_\theta)_+)$, but can be easily extended to $\xi \neq 0$ with $\xi/|\xi| \in \vec{n}((\partial\mathcal{B}_\theta)_+)$ by using the homogeneity of H_θ . We will follow Müller's method used to prove his [20, Lemma 3].

Lemma 3.4. *If $d \geq 3$, for every $\xi \in \vec{n}((\partial\mathcal{B}_\theta)_+)$ there exist d linearly independent vectors $v_l = v_l^\theta(\xi) \in \mathbb{Z}^d$ ($l = 1, \dots, d$) such that*

$$(3.1) \quad \begin{aligned} |v_1| &\asymp (K_\xi^\theta)^{-d-2q-8+1/(d-1)}, \\ |v_l| &\asymp (K_\xi^\theta)^{-d-2q-5+1/(d-1)} \quad (l = 2, \dots, d), \\ |\det(V)| &\asymp (K_\xi^\theta)^{d(-d-2q-5+1/(d-1))}, \\ \|V^{-1}\| &\lesssim (K_\xi^\theta)^{d+2q+2-1/(d-1)}, \end{aligned}$$

where $V = (v_1, \dots, v_d)$. Furthermore there exists a constant $c_2 > 0$ (depending only on q and \mathcal{B}) such that, for $\eta \in B(\xi, c_2(K_\xi^\theta)^{d+2q+7-1/(d-1)})$,

$$(3.2) \quad |h_q^\theta(\eta, v_1, \dots, v_d)| \gtrsim (K_\xi^\theta)^{(-d-2q-5+1/(d-1))d(q+2)-3d+5-1/(d-1)}.$$

The constants implicit in (3.1) and (3.2) depend only on q and \mathcal{B} .

Proof. Let $\xi \in \vec{n}((\partial\mathcal{B}_\theta)_+)$ be arbitrarily fixed.

Step 1. Let $p_1 = \xi$. We first choose $d-1$ vectors $p_2, \dots, p_d \in S^{d-1}$ such that $P = (p_1, \dots, p_d) \in \mathbb{R}^{d \times d}$ is an orthogonal matrix. Let $\tilde{H}_\theta(y) = H_\theta(Py)$. Then \tilde{H}_θ is positively homogeneous of degree one and smooth at e_1 . Since the matrix $\nabla^2 \tilde{H}_\theta(e_1)$ is similar to $\nabla^2 H_\theta(\xi)$ it follows from Lemma 3.3 that the eigenvalues of $\nabla^2 \tilde{H}_\theta(e_1)$ are $0, \beta_1, \dots, \beta_{d-1}$, where $\{\beta_j^{-1}\}_{j=1}^{d-1}$ are the principle curvatures of $\partial\mathcal{B}_\theta$ at $x^\theta(\xi)$. Without loss of generality we assume $\beta_1 = \max_{1 \leq j \leq d-1} \beta_j$, therefore $\beta_1 \geq (K_\xi^\theta)^{-1/(d-1)}$.

Set $A = \nabla^2 \tilde{H}_\theta(e_1)$. A is a symmetric matrix of rank $d-1$ with vanishing first row and column (due to the homogeneity of \tilde{H}_θ ; see the proof of Müller's [20, Lemma 3]). Choose a system of orthonormal eigenvectors w'_1, \dots, w'_{d-1} of A , whose first components vanish, such that the eigenvalue of w'_j is β_j . For $\alpha > 1$ denote

$$w_l = \begin{cases} w'_1 + \alpha e_1 & \text{if } l = 1, \\ w'_l & \text{if } 2 \leq l \leq d-1, \\ e_1 & \text{if } l = d. \end{cases}$$

Then $Aw_l = \beta_l w'_l$ ($l = 1, \dots, d-1$) and w_1 is orthogonal to w'_l ($l = 2, \dots, d-1$). We also have $|w_1| \asymp \alpha$, $|w_l| = 1$ ($l = 2, \dots, d$), and $|\det(W)| = 1$ where $W = (w_1, \dots, w_d)$. Let $v_l^* = Pw_l$. Then $|v_1^*| \asymp \alpha$, $|v_l^*| = 1$ ($l = 2, \dots, d$), and $|\det(V^*)| = 1$ where $V^* = (v_1^*, \dots, v_d^*)$. We claim that if $\alpha = C_{q,\mathcal{B}}(K_\xi^\theta)^{-3}$ with a sufficiently large $C_{q,\mathcal{B}}$ then

$$(3.3) \quad |h_q^\theta(\xi, v_1^*, \dots, v_d^*)| \gtrsim (K_\xi^\theta)^{-3d+5-1/(d-1)}$$

with $F_\theta(u_1, \dots, u_d) = H_\theta(\xi + \sum_{l=1}^d u_l v_l^*)$.

This claim can be proved by a straightforward computation (given below). Note that $F_\theta(u_1, \dots, u_d) = \tilde{H}_\theta(e_1 + \sum_{l=1}^d u_l w_l)$ and we will use this formula to compute $g_{i,j}^\theta(\xi, v_1^*, \dots, v_d^*) =: b_{i,j}^\theta(\alpha)$. If $1 \leq i, j \leq d-1$,

$$(3.4) \quad b_{i,j}^\theta(0) = (\nabla \cdot w'_1)(\nabla \cdot w'_i)(\nabla \cdot w'_j) \partial_{y_1}^{q-1} \tilde{H}_\theta(e_1) \lesssim (K_\xi^\theta)^{-3}.$$

The last inequality is due to the homogeneity of \tilde{H}_θ (see the proof of Müller's [20, Lemma 3]) and Remark 3.2.

If $i = 1, 1 \leq j \leq d-1$, then

$$(3.5) \quad b_{1,j}^\theta(\alpha) = b_{1,j}^\theta(0) + 3\alpha(-1)^q q! \beta_1 \delta_{1j},$$

where δ_{ij} is the Kronecker notation.

If $2 \leq i, j \leq d-1$, then

$$(3.6) \quad b_{i,j}^\theta(\alpha) = b_{i,j}^\theta(0) + \alpha(-1)^q q! \beta_j \delta_{ij}.$$

If $1 \leq i \leq d, j = d$, then

$$(3.7) \quad b_{i,d}^\theta(\alpha) = (-1)^q q! \beta_1 \delta_{1i}.$$

Using formulas (3.6) and (3.7), we get

$$\begin{aligned} |h_q^\theta(\xi, v_1^*, \dots, v_d^*)| &= (q! \beta_1)^2 |\det(b_{i,j}^\theta(\alpha))_{2 \leq i, j \leq d-1}| \\ &= (q! \beta_1)^2 |\det(b_{i,j}^\theta(0) + \alpha(-1)^q q! \beta_j \delta_{ij})_{2 \leq i, j \leq d-1}| \\ &= \beta_1 (K_\xi^\theta)^{-3d+5} |q!^d C_{q,B}^{d-2} + O(C_{q,B}^{d-3})| \end{aligned}$$

where we have used (3.4), $\beta_j \gtrsim 1$, and $\prod \beta_j = (K_\xi^\theta)^{-1}$ to get the last equality. Since $\beta_1 \geq (K_\xi^\theta)^{-1/(d-1)}$, we get (3.3) if $C_{q,B}$ is sufficiently large.

Step 2. For any $N \in \mathbb{N}$, there exist $v_l \in \mathbb{Z}^d$ ($l = 1, \dots, d$) such that $|v_l^{**} - v_l^*| \leq \sqrt{d}/N$ where $v_l^{**} = v_l/N$. If $N \geq C(K_\xi^\theta)^{-3}$ then $|v_1^{**}| \asymp (K_\xi^\theta)^{-3}$, $|v_l^{**}| \asymp 1$ ($l = 2, \dots, d$), and $|\det(V^{**})| \asymp 1$ where $V^{**} = (v_1^{**}, \dots, v_d^{**})$.

Assume N is the smallest integer not less than $C'(K_\xi^\theta)^{-d-2q-5+1/(d-1)}$ with C' chosen below and $\eta \in B(\xi, c_2 r^\theta(\xi))$ with $r^\theta(\xi) = (K_\xi^\theta)^{d+2q+7-1/(d-1)}$ and $c_2 \leq c_1$, where c_1 is the constant appearing in Lemma 2.1. By the mean value theorem, Lemma 2.1, and Remark 3.2, we get

$$\begin{aligned} &|g_{i,j}^\theta(\xi, v_1^*, \dots, v_d^*) - g_{i,j}^\theta(\eta, v_1^{**}, \dots, v_d^{**})| \\ &\lesssim \begin{cases} (K_\xi^\theta)^{-2q-10}(N^{-1} + c_2(K_\xi^\theta)^{-2}r^\theta(\xi)) & \text{if } i = j = 1, \\ (K_\xi^\theta)^{-2q-7}(N^{-1} + c_2(K_\xi^\theta)^{-2}r^\theta(\xi)) & \text{if } i = 1, j \geq 2, \\ (K_\xi^\theta)^{-2q-4}(N^{-1} + c_2(K_\xi^\theta)^{-2}r^\theta(\xi)) & \text{if } i \geq 2, j \geq 2. \end{cases} \end{aligned}$$

These estimates, together with the bounds of $g_{i,j}^\theta(\xi, v_1^*, \dots, v_d^*)$'s (given by (3.4), (3.5), (3.6), and (3.7)), lead to

$$|h_q^\theta(\xi, v_1^*, \dots, v_d^*) - h_q^\theta(\eta, v_1^{**}, \dots, v_d^{**})| \lesssim (K_\xi^\theta)^{-4d-2q}(N^{-1} + c_2(K_\xi^\theta)^{-2}r^\theta(\xi)).$$

If C' is sufficiently large and c_2 is sufficiently small, it then follows from (3.3) that

$$|h_q^\theta(\eta, v_1^{**}, \dots, v_d^{**})| \gtrsim (K_\xi^\theta)^{-3d+5-1/(d-1)}.$$

The desired estimates now follow from the following two equalities:

$$|h_q^\theta(\eta, v_1, \dots, v_d)| = N^{d(q+2)} |h_q^\theta(\eta, v_1^{**}, \dots, v_d^{**})|$$

and

$$V^{-1} = N^{-1}(\text{adjugate matrix of } V^{**})/\det(V^{**}).$$

□

For $d = 2$ case Guo's [5, Lemma 3.4] gives a similar result but in a nicer form. That lemma can be proved by using the same method. In particular, the bound $g_{11}(\xi, v_1, v_2) \lesssim (K_\xi^\theta)^{-2q-1}$ is used in its proof, but later we find a better bound of g_{11} , namely $g_{11}(\xi, v_1, v_2) \lesssim (K_\xi^\theta)^{-3}$ (just like the bound (3.4) in the above proof). By using the latter bound without modifying too much of the proof of Guo's [5, Lemma 3.4], we are able to prove the following improved result, which eventually leads to our estimates in Theorem 1.4 and 1.5.

Lemma 3.5. *If $d = 2$, for every $\xi \in \vec{n}((\partial\mathcal{B}_\theta)_+)$ there exist two orthogonal vectors $v_i = v_i(\xi) \in \mathbb{Z}^2$ ($i = 1, 2$) such that*

$$(3.8) \quad |v_1| = |v_2| \asymp (K_\xi^\theta)^{-2q-2} \quad \text{and} \quad \|V^{-1}\| \lesssim (K_\xi^\theta)^{2q+2},$$

where $V = (v_1, v_2)$. Furthermore there exists a constant $c_2 > 0$ (depending only on q and \mathcal{B}) such that, for $\eta \in B(\xi, c_2(K_\xi^\theta)^{2q+4})$,

$$(3.9) \quad |h_q^\theta(\eta, v_1, v_2)| \gtrsim (K_\xi^\theta)^{-4q^2-12q-10}.$$

The constants implicit in (3.8) and (3.9) depend only on q and \mathcal{B} .

4. THE FOURIER TRANSFORM OF CERTAIN INDICATOR FUNCTIONS

In this section we will establish an asymptotic formula of the Fourier transform of the indicator function $\chi_{\mathcal{B}}$ for convex domains \mathcal{B} in \mathbb{R}^d , which generalizes the results in Guo's [5, Section 4].

Lemma 4.1. *Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$). Then there exist two positive constants c and c_3 (both depending only on \mathcal{B}) such that, for any $\xi \in \vec{n}((\partial\mathcal{B})_+) \cap (-\vec{n}((\partial\mathcal{B})_+))$ and $r \leq c_3$,*

$$(4.1) \quad |\langle \vec{n}(x), \xi \rangle| \leq 1 - cr^2(\min(K_\xi, K_{-\xi}))^4$$

if x is in $\partial\mathcal{B} \setminus (B(x(\xi), rK_\xi) \cup B(x(-\xi), rK_{-\xi}))$.

Proof. It follows from Lemma A.1 that there exists a constant $c_3 > 0$ (depending only on \mathcal{B}) such that, for any $r \leq c_3$, the Gauss map is bijective from $B(x(\xi), rK_\xi) \cap \partial\mathcal{B}$ and $B(x(-\xi), rK_{-\xi}) \cap \partial\mathcal{B}$ to two subsets of S^{d-1} which contain $B(\xi, c'r(K_\xi)^2) \cap S^{d-1}$ and $B(-\xi, c'r(K_{-\xi})^2) \cap S^{d-1}$ respectively where the constant $c' > 0$ depends only on \mathcal{B} . Then the lemma follows easily with $c = 2c'^2/\pi^2$. \square

Theorem 4.2. *Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$). Let n_l ($l = 1, \dots, d$) be the l^{th} component of the Gauss map of $\partial\mathcal{B}$ and dS the induced Lebesgue*

measure on surfaces (curves if $d = 2$). For any $\xi \in \vec{n}((\partial\mathcal{B})_+) \cap (-\vec{n}((\partial\mathcal{B})_+))$ we have

$$\begin{aligned} \widehat{n_l dS}(\lambda\xi) &= (e^{\pi i(d-1)/4} \xi_l (K_\xi)^{-1/2} e^{-2\pi i \lambda H(\xi)} \\ &\quad + e^{-\pi i(d-1)/4} (-\xi_l) (K_{-\xi})^{-1/2} e^{2\pi i \lambda H(-\xi)}) \lambda^{-(d-1)/2} \\ &\quad + O(\lambda^{-(d+1)/2} \delta^{-(d+5)/2} + \lambda^{-N} \delta^{-4N}) \quad \text{for } \lambda > 0, \end{aligned}$$

where $H(\xi) = \sup_{y \in \mathcal{B}} \langle y, \xi \rangle$, $N \in \mathbb{N}$, and $\delta = \min(K_\xi, K_{-\xi})$. The implicit constant depends only on N and \mathcal{B} .

Proof. We will only prove the case $d \geq 3$ below while the case $d = 2$ is easier and can be handled in the same way. Note that there exists a $C_0 > 0$ such that, for any $x \in \partial\mathcal{B}$, the boundary $\partial\mathcal{B}$ in a neighborhood of x can be parametrized by

$$(4.2) \quad \begin{aligned} \vec{r}(u, x) &= x + \sum_{j=1}^{d-1} u_j \vec{t}_j(x) + h(u, x)(-\vec{n}(x)), \\ \text{for } u &= (u_1, \dots, u_{d-1}) \in B_0 = B(0, C_0) \subset \mathbb{R}^{d-1}, \end{aligned}$$

where $\{\vec{t}_j(x)\}_{j=1}^{d-1}$ is an orthonormal basis of the tangent plane of $\partial\mathcal{B}$ at x (we require that the basis $\{\vec{t}_1(x), \dots, \vec{t}_{d-1}(x), -\vec{n}(x)\}$ has the same orientation as $\{e_1, \dots, e_d\}$) and $h(\cdot, x) \in C^\infty(B_0)$ such that $h(0, x) = 0$, $\nabla_u h(0, x) = 0$, and $\det \nabla_{uu}^2 h(0, x) = K(x)$.

For any fixed $\xi \in \vec{n}((\partial\mathcal{B})_+) \cap (-\vec{n}((\partial\mathcal{B})_+))$ decompose n_l as a sum

$$n_l = \psi_1 + \psi_2 + \psi_3$$

where

$$\psi_1(x, \xi) = n_l(x) \chi_0\left(\frac{x - x(\xi)}{c_4 K_\xi}\right) \text{ and } \psi_2(x, \xi) = n_l(x) \chi_0\left(\frac{x - x(-\xi)}{c_4 K_{-\xi}}\right),$$

where $c_4 > 0$ is determined below and χ_0 is the fixed cut-off function (see §1).

We first estimate $\widehat{\psi_1 dS}$ (while $\widehat{\psi_2 dS}$ is handled in the same way). Applying the parametrization (4.2) at $x(\xi)$ yields

$$(4.3) \quad \widehat{\psi_1 dS}(\lambda\xi) = e^{-2\pi i \lambda \langle \xi, x(\xi) \rangle} \int \tau(u, \xi) e^{2\pi i \lambda h(u, x(\xi))} du,$$

where $\tau(u, \xi) = \psi_1(\vec{r}(u, x(\xi)), \xi) (1 + |\nabla_u h(u, x(\xi))|^2)^{1/2}$ such that

$$\tau(\cdot, \xi) \in C_c^\infty(B(0, c_4 K_\xi))$$

and

$$|D_u^\nu \tau(u, \xi)| \leq C(c_4 K_\xi)^{-|\nu|}.$$

By a change of variable the integral in (4.3), denoted by $\Delta(\xi)$, is

$$\Delta(\xi) = K_\xi^{d-1} \int \tau(K_\xi u, \xi) e^{2\pi i \lambda h(K_\xi u, x(\xi))} du.$$

Applying a quantitative version of the Morse Lemma (see the proof of Sogge and Stein's [26, Lemma 2]) we can find an $\alpha_1 > 0$ and a smooth invertible mapping $u \mapsto v$ from $B(0, \alpha_1)$ to a neighborhood of the origin in v -space, so that $|D_u^\nu v| \leq C$, $|D_v^\nu u| \leq C$, $\det(\nabla_v u(0)) = 1$, and

$$h(K_\xi u, x(\xi)) = K_\xi^2(\mu_1 v_1^2 + \dots + \mu_{d-1} v_{d-1}^2)/2, \quad u \in B(0, \alpha_1),$$

where μ_1, \dots, μ_{d-1} are the eigenvalues of the matrix $\nabla_{uu}^2 h(0, x(\xi))$. Let $c_4 \leq \alpha_1$. Then

$$\Delta(\xi) = K_\xi^{d-1} \int \tilde{\tau}(v, \xi) e^{i\tilde{\lambda}(\mu_1 v_1^2 + \dots + \mu_{d-1} v_{d-1}^2)/2} dv,$$

where $\tilde{\lambda} = 2\pi\lambda K_\xi^2$ and $\tilde{\tau}(v, \xi) = \tau(K_\xi u(v), \xi) |\det(\nabla_v u)|$. Applying Lemma A.3 to the integral above yields an asymptotic expansion, which in turn gives

$$\begin{aligned} \widehat{\psi_1 dS}(\lambda\xi) &= e^{\pi i(d-1)/4} \xi_l (K_\xi)^{-1/2} e^{-2\pi i\lambda H(\xi)} \lambda^{-(d-1)/2} \\ &\quad + O(\lambda^{-(d+1)/2} K_\xi^{-(d+5)/2}). \end{aligned}$$

The estimate $\widehat{\psi_3 dS} = O(\lambda^{-N} \delta^{-4N})$ follows from Lemma 4.1 and integration by parts (see Stein [27, p. 350] for a similar argument). This finishes the proof. \square

As a consequence of the Gauss–Green formula we get:

Corollary 4.3. *Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 2$) is a compact convex domain and its boundary is a smooth hypersurface (curve if $d = 2$). For any $\xi \in \vec{n}((\partial\mathcal{B})_+) \cap (-\vec{n}((\partial\mathcal{B})_+))$ we have*

$$\begin{aligned} \widehat{\chi_{\mathcal{B}}}(\lambda\xi) &= ((2\pi)^{-1} e^{\pi i(d+1)/4} (K_\xi)^{-1/2} e^{-2\pi i\lambda H(\xi)} \\ &\quad + (2\pi)^{-1} e^{-\pi i(d+1)/4} (K_{-\xi})^{-1/2} e^{2\pi i\lambda H(-\xi)}) \lambda^{-(d+1)/2} \\ &\quad + O(\lambda^{-(d+3)/2} \delta^{-(d+5)/2} + \lambda^{-N-1} \delta^{-4N}) \quad \text{for } \lambda > 0, \end{aligned}$$

where $H(\xi) = \sup_{y \in \mathcal{B}} \langle y, \xi \rangle$, $N \in \mathbb{N}$, and $\delta = \min(K_\xi, K_{-\xi})$. The implicit constant depends only on N and \mathcal{B} .

5. THE \mathbb{R}^d ($d \geq 3$) CASE

By a very standard argument, Theorem 1.2 follows easily from the following lemma (see Guo [5, p. 18], Iosevich [9, p. 26-27], or Iosevich, Sawyer, and Seeger [10, p. 168-169] for this argument).

Lemma 5.1. *Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 3$) be a compact convex domain and $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \rho(y) dy = 1$. If the boundary is a smooth hypersurface of finite type ω then, for $j \in \mathbb{N}$, we have*

$$(5.1) \quad \int_{SO(d)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{2d/(d+1) + \zeta_d + \sigma(d, \omega)} \left| \sum_{k \in \mathbb{Z}_*^d} \widehat{\chi_{\mathcal{B}_\theta}}(tk) \widehat{\rho}(\varepsilon k) \right| d\theta \lesssim 1,$$

where $d\theta$ is the normalized Haar measure on $SO(d)$, ζ_d and $\sigma(d, \omega)$ are given by (1.3) and (1.4) respectively, and

$$\varepsilon = \varepsilon(j, d, \omega) = 2^{-j\alpha(d, \omega)},$$

$$\alpha(d, \omega) = \begin{cases} 1 - 2[6(\omega - 2)d^4 + 112(\omega - 2)d^3 - 4(\omega - 2)d^2 \\ \quad + (410 - 203\omega)d + 82\omega - 156]/\square & \text{for } 3 \leq d \leq 4, \\ 1 - [4(\omega - 2)d^4 + 90(\omega - 2)d^3 + 61(\omega - 2)d^2 \\ \quad - (227\omega - 456)d + 60\omega - 112]/\triangle & \text{for } d \geq 5, \end{cases}$$

with \square and \triangle given by (1.5) and (1.6) respectively.

Proof. Let $t \in [2^{j-1}, 2^{j+2}]$ and $\delta = \delta(j, d, \omega) = 2^{-j\beta(d, \omega)}$ with

$$\beta(d, \omega) = \begin{cases} 2(\omega - 2)d(d - 1)(d - 2)/\square & \text{for } 3 \leq d \leq 4, \\ (\omega - 2)d(d - 1)(d - 3)/\triangle & \text{for } d \geq 5. \end{cases}$$

For any $\theta \in SO(d)$ we have the following splitting

$$\sum_{k \in \mathbb{Z}_*^d} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon k) = \text{Sum I}(t, \varepsilon, \delta, \theta) + \text{Sum II}(t, \varepsilon, \delta, \theta),$$

where

$$\text{Sum I}(t, \varepsilon, \delta, \theta) = \sum_{k \in D_1(\delta, \theta)} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon k),$$

$$\text{Sum II}(t, \varepsilon, \delta, \theta) = \sum_{k \in D_2(\delta, \theta)} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon k),$$

and $D_1(\delta, \theta)$, $D_2(\delta, \theta)$ are two regions defined as follows:

$$D_2(\delta, 0) = \{\xi \in \mathbb{R}_*^d : \xi/|\xi| \text{ or } -\xi/|\xi| \in \vec{n}(\{x \in \partial\mathcal{B} : K(x) < \delta\})\},$$

$$D_1(\delta, 0) = \mathbb{R}_*^d \setminus D_2(\delta, 0), \quad D_1(\delta, \theta) = \theta D_1(\delta, 0), \quad \text{and } D_2(\delta, \theta) = \theta D_2(\delta, 0).$$

The estimate (5.1) follows from the next two claims. Notice that the finite type condition is only used in the estimate of Sum II and the size estimate of $|D_2(\delta, 0)|$.

Claim 5.2.

$$\int_{SO(d)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{2d/(d+1) + \zeta_d + \sigma(d, \omega)} |\text{Sum II}(t, \varepsilon, \delta, \theta)| d\theta \lesssim 1$$

with an implicit constant depending only on \mathcal{B} .

Claim 5.3.

$$\int_{SO(d)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{2d/(d+1) + \zeta_d + \sigma(d, \omega)} |\text{Sum I}(t, \varepsilon, \delta, \theta)| d\theta \lesssim 1$$

with an implicit constant depending only on \mathcal{B} .

□

Proof of Claim 5.2.

$$\text{L.H.S.} \lesssim (2^j)^{2d/(d+1)-(d+1)/2+\zeta_d+\sigma(d,\omega)} \sum_{k \in \mathbb{Z}_*^d} |k|^{-(d+1)/2} |\widehat{\rho}(\varepsilon k)| (*),$$

where

$$(*) := \int_{SO(d)} 1_{D_2(\delta, \theta)}(k) \left(\sup_{2^{j-1} \leq t \leq 2^{j+2}} |tk|^{(d+1)/2} |\widehat{\chi}_{\mathcal{B}_\theta}(tk)| \right) d\theta.$$

Recalling the definition (1.8) of the function Φ , we get

$$\begin{aligned} (*) &\lesssim \int_{S^{d-1} \cap D_2(\delta, 0)} \Phi(\xi) d\xi, \\ &\lesssim \int_{S^{d-1} \cap D_2(\delta, 0)} (K_\xi)^{-1/2} + (K_{-\xi})^{-1/2} + 1 d\xi, \\ &\lesssim \int_{\overline{n}(\{x \in \partial \mathcal{B}: K(x) < \delta\})} (K_\xi)^{-1/2} d\xi, \\ &\lesssim \int_{\{x \in \partial \mathcal{B}: K(x) < \delta\}} K(x)^{1/2} dA(x), \\ &\lesssim \delta^{1/2+1/(d-1)(\omega-2)}. \end{aligned}$$

In the above estimate of $(*)$ we have used Svensson's estimate of $\Phi(\xi)$ for finite type domains (see [28, p. 19]), the symmetry of $D_2(\delta, 0)$, a change of variables, and Lemma 2.3. Hence

$$\text{L.H.S.} \lesssim (2^j)^{2d/(d+1)-(d+1)/2+\zeta_d+\sigma(d,\omega)} \delta^{1/2+1/(d-1)(\omega-2)} \varepsilon^{-(d-1)/2} \lesssim 1.$$

□

In order to prove Claim 5.3 we need an estimate of d -dimensional exponential sums, which will be included in the appendix for interested readers.

Proof of Claim 5.3. Note if $\xi \in D_1(\delta, \theta)$ then $K_{\pm\xi}^\theta \geq \delta$. Applying Corollary 4.3 to Sum I yields

$$(5.2) \quad \text{Sum I}(t, \varepsilon, \delta, \theta) = (2\pi)^{-1} e^{\pi i(d+1)/4} S_1 + (2\pi)^{-1} e^{-\pi i(d+1)/4} \widetilde{S}_1 + R_1,$$

where

$$S_1(t, \varepsilon, \delta, \theta) = t^{-(d+1)/2} \sum_{k \in D_1(\delta, \theta)} |k|^{-(d+1)/2} (K_k^\theta)^{-1/2} \widehat{\rho}(\varepsilon k) e(tH_\theta(k)),$$

$$\widetilde{S}_1(t, \varepsilon, \delta, \theta) = t^{-(d+1)/2} \sum_{k \in D_1(\delta, \theta)} |k|^{-(d+1)/2} (K_{-k}^\theta)^{-1/2} \widehat{\rho}(\varepsilon k) e(-tH_\theta(-k)),$$

and

$$(5.3) \quad R_1 \lesssim \delta^{-2(d+1)} t^{-(d+3)/2} (\varepsilon^{-(d-3)/2} + \log(\varepsilon^{-1})) \lesssim t^{-2d/(d+1)-\zeta_d-\sigma(d,\omega)}.$$

We will only estimate S_1 since \widetilde{S}_1 is similar. Denote $\mathcal{C}_1 = \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}$. Let us introduce a dyadic decomposition and a partition of unity.

Assume $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a real radial function such that $\text{supp}(\varphi) \subset \mathcal{C}_1$, $0 \leq \varphi \leq 1$, and

$$\sum_{l_0=-\infty}^{\infty} \varphi\left(\frac{\xi}{2^{l_0}}\right) = 1 \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

Denote

$$S_{1,M} = \sum_{k \in D_1(\delta, \theta)} \varphi(M^{-1}k) |k|^{-(d+1)/2} (K_k^\theta)^{-1/2} \widehat{\rho}(\varepsilon k) e(tH_\theta(k)),$$

then

$$(5.4) \quad S_1 = t^{-(d+1)/2} \sum_{l_0=0}^{\infty} S_{1,2^{l_0}}.$$

We will estimate $S_{1,M}$ for a fixed $M = 2^{l_0}$, $l_0 \in \mathbb{N}_0$. Let $q \in \mathbb{N}$. For each $\xi \in \vec{n}((\partial\mathcal{B})_+)$ there exists a cone

$$\mathfrak{C}(\xi, 2r(\xi)) := \bigcup_{l>0} lB(\xi, 2r(\xi)) \subset \mathbb{R}^d,$$

where $r(\xi) = c_2(K_\xi)^{d+2q+7-1/(d-1)}/2$ and c_2 is the constant appearing in the statement of Lemma 3.4. Note that Lemma 2.1 implies that $K_\eta \asymp K_\xi$ if $\eta \in \mathfrak{C}(\xi, 2r(\xi))$. From the family of cones $\{\mathfrak{C}(\xi, r(\xi)/2) : \xi \in \vec{n}((\partial\mathcal{B})_+)\}$, we can choose, by a Vitali procedure, a sequence $\{\mathfrak{C}(\xi_i, r(\xi_i)/2)\}_{i=1}^\infty$ such that these cones still cover $\vec{n}((\partial\mathcal{B})_+)$ and that $\{\mathfrak{C}(\xi_i, r(\xi_i))\}_{i=1}^\infty$ satisfies the bounded overlap property. Denote

$$\mathfrak{C}_i^\theta = \theta \mathfrak{C}(\xi_i, r(\xi_i)).$$

Then the collection $\{\mathfrak{C}_i^\theta\}_{i=1}^\infty$ forms an open cover of $\vec{n}((\partial\mathcal{B}_\theta)_+)$. We can construct a partition of unity $\{\psi_i\}_{i=1}^\infty$ such that

- (i) $\sum_i \psi_i \equiv 1$ on $\vec{n}((\partial\mathcal{B}_\theta)_+)$, and $\psi_i \in C_0^\infty(\mathfrak{C}_i^\theta)$;
- (ii) each ψ_i is positively homogeneous of degree zero;
- (iii) $|D^\nu \psi_i| \lesssim_{|\nu|} (K_{\xi_i})^{-(d+2q+7-1/(d-1))|\nu|}$ on \mathcal{C}_1 .

From the family $\{\mathfrak{C}_i^\theta\}_{i=1}^\infty$ we can find a subfamily $\{\mathfrak{C}_i^\theta\}_{i \in \mathcal{A}}$ which covers $D_1(\delta, \theta)$, where $\mathcal{A} = \mathcal{A}(\delta)$ is an index set such that $i \in \mathcal{A}$ if and only if \mathfrak{C}_i^θ intersects $D_1(\delta, \theta)$. Since $r(\xi_i) \gtrsim \delta^{d+2q+7-1/(d-1)}$ for any $i \in \mathcal{A}$, a size estimate gives that $\#\mathcal{A} \lesssim \delta^{-(d+2q+7-1/(d-1))(d-1)}$. Define

$$(5.5) \quad S_{1,M}^* = \sum_{i \in \mathcal{A}} S_{2,i},$$

where

$$S_{2,i} = \sum_{k \in \mathbb{Z}^d} U_i^\theta(k) e(tH_\theta(k))$$

and

$$U_i^\theta(k) = \psi_i(M^{-1}k) \varphi(M^{-1}k) |k|^{-(d+1)/2} (K_k^\theta)^{-1/2} \widehat{\rho}(\varepsilon k).$$

Instead of $S_{1,M}$ we will estimate $S_{1,M}^*$. It turns out that the error

$$(5.6) \quad R_{2,M} = S_{1,M} - S_{1,M}^*$$

is relatively small and this will be clear at the end of this proof.

To estimate $S_{1,M}^*$ we will estimate $S_{2,i}$ for any fixed $i \in \mathcal{A}$. By Lemma 3.4 and the homogeneity of H_θ , there exist d linearly independent vectors $v_j = v_j(\theta\xi_i) \in \mathbb{Z}^d$ ($j = 1, \dots, d$) such that if $\eta \in \cup_{1/4 \leq l \leq 4} lB(\theta\xi_i, 2r(\xi_i))$ then

$$(5.7) \quad |h_q^\theta(\eta, v_1, \dots, v_d)| \gtrsim (K_{\xi_i})^{(-d-2q-5+1/(d-1))d(q+2)-3d+5-1/(d-1)}.$$

Let $L = [\mathbb{Z}^d : \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \dots \oplus \mathbb{Z}v_d]$ be the index of the lattice spanned by v_1, \dots, v_d in the lattice \mathbb{Z}^d . Then there exist vectors $b_l \in \mathbb{Z}^d$ ($l = 1, \dots, L$) such that

$$\mathbb{Z}^d = \biguplus_{l=1}^L (\mathbb{Z}v_1 + \dots + \mathbb{Z}v_d + b_l).$$

It follows from Lemma 3.4 that

$$L = |\det(v_1, \dots, v_d)| \asymp (K_{\xi_i})^{d(-d-2q-5+1/(d-1))}$$

and

$$|b_l| \lesssim (K_{\xi_i})^{-d-2q-8+1/(d-1)}.$$

Let $N > d/2$ be an arbitrarily fixed natural number. We have

$$(5.8) \quad \begin{aligned} S_{2,i} &= \sum_{l=1}^L \sum_{m \in \mathbb{Z}^d} U_i^\theta \left(\sum_{j=1}^d m_j v_j + b_l \right) e(tH_\theta(\sum_{j=1}^d m_j v_j + b_l)) \\ &= (K_{\xi_i})^{-1/2} M^{-(d+1)/2} (1 + M\varepsilon)^{-N} \sum_{l=1}^L S(T, M_*; G_l, F_l), \end{aligned}$$

where $T = tM$, $M_* = (K_{\xi_i})^{d+2q+2-1/(d-1)} M$,

$$G_l(y) = (K_{\xi_i})^{1/2} M^{(d+1)/2} (1 + M\varepsilon)^N U_i^\theta(M_* V y + b_l),$$

and

$$F_l(y) = H_\theta(M^{-1}(M_* V y + b_l)),$$

where $V = (v_1, \dots, v_d)$.

We consider the function F_l restricted to the convex domain

$$\Omega_l = \{y \in \mathbb{R}^d : M^{-1}(M_* V y + b_l) \in \bigcup_{1/4 \leq l \leq 4} lB(\theta\xi_i, 2r(\xi_i))\}.$$

The support of G_l satisfies

$$\text{supp}(G_l) \subset \{y \in \mathbb{R}^d : M^{-1}(M_* V y + b_l) \in \overline{\mathcal{C}_1 \cap \mathcal{C}_i^\theta}\} \subset \Omega_l.$$

We apply to $S(T, M_*; G_l, F_l)$ Proposition B.1 with $G = G_l$, $F = F_l$, $K = K_{\xi_i}$, and $\Omega = \Omega_l$. And we only compute below the case $d \geq 5$ with $q = 1$ (while the case $3 \leq d \leq 4$ with $q = 2$ can be handled in the same way).

Since $1 \gtrsim K_{\xi_i} \gtrsim \delta$ if $i \in \mathcal{A}$, there exist positive constants C_2 and C_3 such that the assumptions of Proposition B.1 are satisfied if $M \in \mathcal{J}_1$ where \mathcal{J}_1 is an interval defined by

$$\mathcal{J}_1 = [C_3 \delta^{-37d-41+5/(d-1)}, C_2 \delta^{(14d^4+66d^3+61d^2-144d-12)/(2(d-2)(d-1)d)} t^{d/(d-2)}].$$

This follows from Lemma 3.4, (5.7) and the following facts: if

$$(K_{\xi_i})^{-d-2q-8+1/(d-1)} \lesssim M$$

then $\Omega_l \subset c_0 B(0, 1)$ for a constant c_0 (depending only on q, \mathcal{B});

$$\text{dist} \left(\left(\bigcup_{1/4 \leq l \leq 4} lB(\theta \xi_i, 2r(\xi_i)) \right)^c, \overline{\mathcal{C}_1 \cap \mathfrak{C}_i^\theta} \right) \geq c_2 (K_{\xi_i})^{d+2q+7-1/(d-1)}/8;$$

and

$$D^\nu U_i^\theta \lesssim (K_{\xi_i})^{-(d+2q+7-1/(d-1))|\nu|-1/2} M^{-|\nu|-(d+1)/2} (1 + M\varepsilon)^{-N}.$$

Thus by Proposition B.1 we get

$$(5.9) \quad S(T, M_*; G_l, F_l) \lesssim (K_{\xi_i})^{d^2-13d/2-6+9/(d+2)} t^{d/(2d+4)} M^{d-d/(d+2)}.$$

Then by using (5.5), (5.8), (5.9), $K_{\xi_i} \gtrsim \delta$, and bounds of $\#\mathcal{A}$ and L , we get

$$(5.10) \quad S_{1,M}^* \lesssim \delta^{-d^2-43d/2+9/2+1/(d-1)+9/(d+2)} t^{d/(2d+4)} M^{(d-1)/2-d/(d+2)} (1 + M\varepsilon)^{-N}.$$

Now we can estimate S_1 . By (5.4) and (5.6) we get

$$(5.11) \quad S_1 = t^{-(d+1)/2} \left(\sum_{l_0 \in \{n \in \mathbb{N}_0: 2^n \in \mathcal{J}_1\}} S_{1,2^{l_0}}^* + R_2 + R_3 \right),$$

where

$$R_2 = \sum_{l_0 \in \{n \in \mathbb{N}_0: 2^n \in \mathcal{J}_1\}} R_{2,2^{l_0}} \quad \text{and} \quad R_3 = \sum_{l_0 \in \{n \in \mathbb{N}_0: 2^n \in \mathcal{J}_1^c\}} S_{1,2^{l_0}}.$$

Using the bound (5.10) of $S_{1,M}^*$ we get

$$(5.12) \quad \sum_{l_0 \in \{n \in \mathbb{N}_0: 2^n \in \mathcal{J}_1\}} S_{1,2^{l_0}}^* \lesssim \delta^{-d^2-43d/2+9/2+1/(d-1)+9/(d+2)} t^{d/(2d+4)} \varepsilon^{-(d-1)/2+d/(d+2)}.$$

Hence Claim 5.3 follows from (5.2), (5.3), (5.11), (5.12), sizes of δ and ε , and the following estimates of R_2 ⁽³⁾ and R_3 :

$$(5.13) \quad \int_{SO(d)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{2d/(d+1)+\zeta_d+\sigma(d,\omega)} t^{-(d+1)/2} |R_2| d\theta \lesssim 1;$$

$$(5.14) \quad \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{2d/(d+1)+\zeta_d+\sigma(d,\omega)} t^{-(d+1)/2} |R_3| \lesssim 1.$$

The (5.13) follows from Lemma 2.2 and 2.3 if we notice that

$$|R_{2,M}| \lesssim \delta^{-1/2} \sum_{k \in D_2(\delta, \theta)} |\varphi(M^{-1}k)| |k|^{-(d+1)/2} |\hat{\rho}(\varepsilon k)|;$$

and the (5.14) is true since we have, by trivial estimates,

$$|R_3| \lesssim \delta^{-5d^2-7d/2+9} + \delta^{-1/2} t^{d(d-1)/(2(d-2))-N_1 d/(d-2)} \varepsilon^{-N_1},$$

for any integer $N_1 > (d-1)/2$. □

Just like that Lemma 5.1 implies Theorem 1.2, the following lemma implies Theorem 1.3. Its proof is essentially the same as above, however, we now use, in the estimate of Sum II, Hölder's inequality and Varchenko's [29, Theorem 8] instead of Svensson's estimate of $\Phi(\xi)$ (as we mentioned in §1) and $\delta = t^{-\beta(d,\infty)}$ with $\beta(d,\infty) = \lim_{\omega \rightarrow \infty} \beta(d,\omega)$.

Lemma 5.4. *Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geq 3$) be a compact convex domain and $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \rho(y) dy = 1$. If the boundary is a smooth hypersurface then*

$$\int_{SO(d)} t^{2d/(d+1)+\zeta_d} \left| \sum_{k \in \mathbb{Z}_*^d} \hat{\chi}_{\mathcal{B}_\theta}(tk) \hat{\rho}(\varepsilon k) \right| d\theta \lesssim 1,$$

where $d\theta$ is the normalized Haar measure on $SO(d)$, ζ_d is given by (1.3), and

$$\varepsilon = t^{-\alpha(d,\infty)}$$

with

$$\alpha(d,\infty) = \begin{cases} 1 - \frac{12d^4+224d^3-8d^2-406d+164}{6d^5+118d^4+109d^3-216d^2-119d+82} & \text{for } 3 \leq d \leq 4, \\ 1 - \frac{4d^4+90d^3+61d^2-227d+60}{2d^5+47d^4+76d^3-85d^2-82d+30} & \text{for } d \geq 5. \end{cases}$$

Remark 5.5. Note that $\alpha(d,\infty) = \lim_{\omega \rightarrow \infty} \alpha(d,\omega)$.

³The method used here to estimate R_2 is different from what we used in [5]. More precisely, we estimate the integral of R_2 rather than R_2 itself. Here we need the size estimate of $|D_2(\delta, 0)|$, and this is the only place in the estimate of Sum I where the finite type condition is used.

6. THE \mathbb{R}^2 CASE

To prove Theorem 1.4 and 1.5 the key step is to prove the following \mathbb{R}^2 analogues of Lemma 5.1.

Lemma 6.1. *Let $\zeta_2 = 1/2859$, $\mathcal{B} \subset \mathbb{R}^2$ be a compact convex domain with a smooth boundary, and $\rho \in C_0^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho(y) dy = 1$. Then, for $j \in \mathbb{N}$, we have*

$$\int_{SO(2)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{4/3+\zeta_2} \left| \sum_{k \in \mathbb{Z}_*^2} \hat{\chi}_{\mathcal{B}_\theta}(tk) \hat{\rho}(\varepsilon(j, \infty)k) \right| d\theta \lesssim 1,$$

where $d\theta$ is the normalized Haar measure on $SO(2)$ and $\varepsilon(j, \infty) = 2^{-318j/953}$. Furthermore, if the boundary is of finite type ω then

$$\int_{SO(2)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{4/3+\zeta_2+\sigma(2, \omega)} \left| \sum_{k \in \mathbb{Z}_*^2} \hat{\chi}_{\mathcal{B}_\theta}(tk) \hat{\rho}(\varepsilon(j, \omega)k) \right| d\theta \lesssim 1,$$

where $\varepsilon(j, \omega) = 2^{-j\alpha(2, \omega)}$,

$$\alpha(2, \omega) = \frac{318\omega - 616}{953\omega - 1848} \quad \text{and} \quad \sigma(2, \omega) = \frac{616}{953(953\omega - 1848)}.$$

Since the proof is essentially the same as the proof of Lemma 5.1 we will not provide every detail but only a few key estimates (see also the proof of Guo's [5, Lemma 6.1]).

As before we first decompose $\sum_{k \in \mathbb{Z}_*^2} \hat{\chi}_{\mathcal{B}_\theta}(tk) \hat{\rho}(\varepsilon k)$ into two parts: Sum I and II.

By Lemma 2.2 and the fact that $\Phi \in L^{2, \infty}(S^1)$ ([2, Theorem 0.3]) we get

$$\int_{SO(2)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{4/3+\zeta_2} |\text{Sum II}| d\theta \lesssim (2^j)^{4/3-3/2+\zeta_2} \delta^{1/2} \varepsilon^{-1/2} \lesssim 1,$$

where $\varepsilon = \varepsilon(j, \infty)$ and $\delta = \delta(j, \infty) = 2^{-j/953}$.

If $\partial\mathcal{B}$ is of finite type ω , then

$$\begin{aligned} & \int_{SO(2)} \sup_{2^{j-1} \leq t \leq 2^{j+2}} t^{4/3+\zeta_2+\sigma(2, \omega)} |\text{Sum II}| d\theta \\ & \lesssim (2^j)^{4/3-3/2+\zeta_2+\sigma(2, \omega)} \delta^{1/2+1/(\omega-2)} \varepsilon^{-1/2} \lesssim 1, \end{aligned}$$

where $\varepsilon = \varepsilon(j, \omega)$ and $\delta = \delta(j, \omega) = 2^{-j\beta(2, \omega)}$ with

$$\beta(2, \omega) = \frac{\omega - 2}{953\omega - 1848}.$$

For Sum I we now use Lemma 3.5 and get

$$\text{Sum I} \lesssim \delta^{-14} t^{-3/2+1/22} \varepsilon^{-7/22} + t^{-3/2} |R_2|.$$

Combining this with the above two estimates of Sum II yields Lemma 6.1.

APPENDIX A. SEVERAL LEMMAS

Here is a quantitative version of the inverse function theorem (see the appendix in Guo [4]).

Lemma A.1. *Suppose f is a $C^{(k)}$ ($k \geq 2$) mapping from an open set $\Omega \subset \mathbb{R}^d$ into \mathbb{R}^d and $b = f(a)$ for some $a \in \Omega$. Assume $|\det(\nabla f(a))| \geq c$ and for any $x \in \Omega$,*

$$|D^\nu f_i(x)| \leq C \quad \text{for } |\nu| \leq 2, 1 \leq i \leq d.$$

If $r_0 \leq \sup\{r > 0 : B(a, r) \subset \Omega\}$, then f is bijective from $B(a, r_1)$ to an open set containing $B(b, r_2)$ where

$$r_1 = \min\left\{\frac{c}{2d^2d!C^d}, r_0\right\}, \quad r_2 = \frac{c}{4d!C^{d-1}}r_1.$$

The inverse mapping f^{-1} is in $C^{(k)}(V)$.

Hörmander's [7, Theorem 7.7.1] gives the following estimate obtained by integration by parts.

Lemma A.2. *Let $K \subset \mathbb{R}^d$ be a compact set, X an open neighborhood of K and k a nonnegative integer. If $u \in C_0^k(K)$, real $f \in C^{k+1}(X)$, then*

$$\left| \int u(x) e^{i\lambda f(x)} dx \right| \leq C|K|\lambda^{-k} \sum_{|\nu| \leq k} \sup |D^\nu u| |\nabla f|^{|\nu|-2k}, \quad \lambda > 0.$$

Here C is bounded when f stays in a bounded set in $C^{k+1}(X)$.

The following lemmas are various results of the method of stationary phase. The first one is Hörmander's [7, Lemma 7.7.3]. The second one is Sogge and Stein's [26, Lemma 2].

Lemma A.3. *Let A be a symmetric non-degenerate matrix with $\text{Im} A \geq 0$. Then we have for every integer $k > 0$ and integer $s > d/2$*

$$\begin{aligned} & \left| \int u(x) e^{i\lambda \langle Ax, x \rangle / 2} dx - (\det(\lambda A / 2\pi i))^{-1/2} T_k(\lambda) \right| \\ & \leq C_k (\|A^{-1}\|/\lambda)^{d/2+k} \sum_{|\alpha| \leq 2k+s} \|D^\alpha u\|_{L^2}, \quad u \in \mathcal{S}, \end{aligned}$$

$$T_k(\lambda) = \sum_{j=0}^{k-1} (2i\lambda)^{-j} \langle A^{-1}D, D \rangle^j u(0) / j!.$$

Lemma A.4. *Suppose ϕ and ψ are smooth functions in $B(0, \delta) \subset \mathbb{R}^d$ with ϕ real-valued. Assume that $|(\partial/\partial x)^\nu \phi| \leq C_1$, $|\nu| \leq d+2$ and $|(\partial/\partial x)^\nu \psi| \leq C_2 \delta^{-|\nu|}$, $|\nu| \leq d$. We also suppose that $(\nabla \phi)(0) = 0$, but $|\det \nabla^2 \phi(0)| \geq \delta$.*

Then there exists a positive constant c_1 (independent of δ), which is sufficiently small, so that if ψ is supported in $B(0, c_1\delta)$ we can assert that

$$\left| \int_{\mathbb{R}^d} \psi e^{i\lambda\phi} dx \right| \leq C\lambda^{-d/2}\delta^{-1/2}.$$

APPENDIX B. ESTIMATE OF EXPONENTIAL SUMS

In this section we will prove a higher dimensional analogue of Guo's [5, Proposition 5.2] by using the same method.

Let $M_* > 1$ and $T > 0$ be parameters. We consider d -dimensional exponential sums of the form

$$S(T, M_*; G, F) = \sum_{m \in \mathbb{Z}^d} G(m/M_*) e(TF(m/M_*)),$$

where $G : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth, compactly supported, and bounded above by a constant, and $F : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth on an open convex domain Ω such that

$$\text{supp}(G) \subset \Omega \subset c_0 B(0, 1),$$

where $c_0 > 0$ is a fixed constant.

Proposition B.1. *Let $d \geq 3$, $q \in \mathbb{N}$, $Q = 2^q$, and $K < 1$ be a positive parameter. Assume that*

$$(B.1) \quad \text{dist}(\text{supp}(G), \Omega^c) \gtrsim K^{d+2q+13-1/(d-1)},$$

for all $\nu \in \mathbb{N}_0^d$ and $y \in \Omega$

$$(B.2) \quad D^\nu G(y) \lesssim K^{-(d+2q+13-1/(d-1))|\nu|},$$

$$(B.3) \quad D^\nu F(y) \lesssim \begin{cases} K^{-6|\nu|} & \text{if } 0 \leq |\nu| \leq 1, \\ K^{3-8|\nu|} & \text{if } |\nu| \geq 2, \end{cases}$$

and for $\mu = (1, 0, \dots, 0, q-1) \in \mathbb{N}_0^d$

$$(B.4) \quad |\det(\nabla^2 D^\mu F(y))| \gtrsim K^{-3(q+3)d+5-1/(d-1)}.$$

If

$$(B.5) \quad M_* \geq K^{-4(5q+4)d-37+4/(d-1)}$$

and

$$(B.6) \quad T \geq K^{-I/(Q(d-1)d^2)} M_*^{q+2/Q-2/d}$$

with

$$I = 2(5q+4)(2Q-3)d^4 + (-35+25q+40Q+2qQ)d^3 + (60+5q-Q-17qQ)d^2 + (6-60Q-5qQ)d-6Q,$$

then

$$(B.7) \quad S(T, M_*; G, F) \lesssim [K^{-(2(5q+4)d^3+(3q+19)d^2-(13q+24)d-6)/(d(d-1))} TM_*^{2(Q-1)d+2Q-q-2}]^{d/(2Q+2(Q-1)d)}.$$

The constant implicit in (B.7) depends only on d , q , c_0 , and constants implicit in (B.1), (B.2), (B.3), and (B.4).

Proof. Let H be a parameter satisfying

$$(B.8) \quad 1 < H \leq c_5 K^{(6(5q+4)d^3-5(5q-7)d^2-5(q+12)d-6)/(2(d-1)d)} M_*$$

with $c_5 < 1$ chosen (later) to be sufficiently small. Then $H \leq M_*$. We apply to $S(T, M_*; G, F)$ the iterated one-dimensional Weyl-Van der Corput inequality with $r_1 = e_1$ and $r_j = e_d$ ($j = 2, \dots, q$) (see [4, Lemma 2.2] for this inequality and notations like G_q , F_q , \mathcal{H} , and Ω_q that we will use below). Then we need to estimate $S_4 := S(\mathcal{H}TM_*^{-q}, M_*; G_q, F_q)$. Applying the Poisson summation formula followed by a change of variables yields

$$S_4 = \sum_{p \in \mathbb{Z}^d} K^{8d} M_*^d \int_{\mathbb{R}^d} \Psi_q(z) e(\mathcal{H}TM_*^{-q} F_q(K^8 z) - K^8 M_* \langle p, z \rangle) dz,$$

where $\Psi_q(z) = G_q(K^8 z)$. It is obvious that

$$(B.9) \quad \text{supp}(\Psi_q) \subset K^{-8} \Omega_q \subset c_0 K^{-8} B(0, 1).$$

By (B.1) we also have

$$(B.10) \quad \text{dist}(\text{supp}(\Psi_q), (K^{-8} \Omega_q)^c) \gtrsim K^{d+2q+5-1/(d-1)}.$$

By the assumption (B.3) there exists a constant A_1 such that

$$|\nabla_z(F_q(K^8 z))| \leq (A_1/2) K^{3-8q}.$$

We split S_4 into two parts

$$S_4 = \sum_{|p| < A_1 K^{-8q-5} \mathcal{H}TM_*^{-q-1}} + \sum_{|p| \geq A_1 K^{-8q-5} \mathcal{H}TM_*^{-q-1}} =: S_5 + R_5.$$

It is not hard to prove, by integration by parts (Lemma A.2), that

$$(B.11) \quad R_5 \lesssim K^{-(d+2q+13-1/(d-1))(d+1)} M_*^{-1}.$$

Define $\lambda_1 = K^{3-8q} \mathcal{H}TM_*^{-q}$ and

$$\Phi_q(z, p) = (\mathcal{H}TM_*^{-q} F_q(K^8 z) - K^8 M_* \langle p, z \rangle) / \lambda_1,$$

then

$$(B.12) \quad S_5 = K^{8d} M_*^d \sum_{|p| < A_1 K^{-8} \lambda_1 M_*^{-1}} \int_{\mathbb{R}^d} \Psi_q(z) e(\lambda_1 \Phi_q(z, p)) dz.$$

To estimate S_5 we discuss in two cases.

CASE 1: $\lambda_1 \geq K^{-4(5q+4)d-29+4/(d-1)}$.

For all $z \in K^{-8}\Omega_q$, by (B.2), (B.3), and (B.4), we get

$$(B.13) \quad D_z^\nu \Psi_q(z) \lesssim K^{-(d+2q+5-1/(d-1))|\nu|},$$

$$(B.14) \quad D_z^\nu \Phi_q(z, p) \lesssim \begin{cases} K^{-8} & \text{for } \nu = 0, \\ 1 & \text{for } |\nu| \geq 1, \end{cases}$$

and

$$(B.15) \quad |\det(\nabla_{zz}^2 \Phi_q(z, p))| \gtrsim K^{(5q+4)d+5-1/(d-1)}.$$

To prove this lower bound (B.15) we first note, by using the definition of F_q and the mean value theorem, that for $\mu = (1, 0, \dots, 0, q-1) \in \mathbb{N}_0^d$

$$\frac{\partial^2}{\partial z_{l_1} \partial z_{l_2}}(\Phi_q(z, p)) = K^{8q+13} \frac{\partial^2 D^\mu F}{\partial x_{l_1} \partial x_{l_2}}(K^8 z) + O(K^{-8} \frac{H}{M_*}).$$

The two terms on the right are $\lesssim 1$ and $c_5 K^{(5q+4)d+5-1/(d-1)}$ respectively (implied by (B.3) and (B.8)). Thus

$$\det(\nabla_{zz}^2(\Phi_q(z, p))) = K^{(8q+13)d} \det(\nabla^2 D^\mu F) + O(c_5 K^{(5q+4)d+5-1/(d-1)}).$$

By (B.4), we get (B.15) if we pick a sufficiently small c_5 .

With (B.9), (B.10), (B.13), (B.14), and (B.15), we can estimate the integrals in S_5 . Let us fix an arbitrary $p \in \mathbb{Z}^d$ with $|p| < A_1 K^{-8} \lambda_1 M_*^{-1}$.

We first need to estimate the number of critical points of the phase function $\Phi_q(z, p)$. Denote $\tilde{p} = K^8 M_* p / \lambda_1$ and $F(z) = K^{8q-3} \nabla_z(F_q(K^8 z))$, then $\nabla_z \Phi_q(z, p) = F(z) - \tilde{p}$ and the critical points are determined by the equation

$$F(z) = \tilde{p} \quad \text{for } z \in K^{-8}\Omega_q.$$

The bounds (B.14) and (B.15) imply that the mapping F and its components F_j satisfy

$$D^\nu F_j(z) \lesssim 1 \quad \text{for } |\nu| \leq 2, j = 1, \dots, d,$$

and

$$|\det(\nabla_z F(z))| \gtrsim K^{(5q+4)d+5-1/(d-1)}.$$

By (B.10), we know that $\text{supp}(\Psi_q)$ is strictly smaller than the domain $K^{-8}\Omega_q$ and the distance between their boundary is larger than $a_1 K^{d+2q+5-1/(d-1)}$ for some positive constant a_1 . Let $r_0 = a_1 K^{d+2q+5-1/(d-1)}/2$. By Taylor's formula, there exists a positive constant $a_2 (< a_1/2)$ such that if \tilde{z} is a critical point in $(\text{supp}(\Psi_q))_{(r_0)}$ then, for any $z \in B(\tilde{z}, a_2 K^{d+2q+5-1/(d-1)})$,

$$(B.16) \quad |\nabla_z \Phi_q(z, p)| \gtrsim K^{(5q+4)d+5-1/(d-1)} |z - \tilde{z}|.$$

Applying Lemma A.1 to F with r_0 as above yields two positive constants $a_3 (< a_2/2)$ and a_4 such that if

$$r_1 = a_3 K^{(5q+4)d+5-1/(d-1)} \quad \text{and} \quad r_2 = a_4 K^{2((5q+4)d+5-1/(d-1))},$$

then F is bijective from $B(z, 2r_1)$ to an open set containing $B(F(z), 2r_2)$ for any $z \in (\text{supp}(\Psi_q))_{(r_0)}$. It follows, simply by a size estimate, that the number of critical points in $(\text{supp}(\Psi_q))_{(r_1)}$ is $\lesssim (K^{-8}/r_1)^d \lesssim K^{-((5q+4)d+13-1/(d-1))d}$.

For the p that we have fixed, let Z_j ($j = 1, \dots, J(p)$) be all critical points in $(\text{supp}(\Psi_q))_{(r_1)}$ of the phase function $\Phi_q(z, p)$ and $\chi_j(z) = \chi_0((z - Z_j)/(c_6 r_1))$ with c_6 chosen below. Then the integral in S_5 can be decomposed as

$$(B.17) \quad \int \Psi_q(z) e(\lambda_1 \Phi_q(z, p)) dz = S_6 + R_6,$$

where

$$S_6 = \sum_{j=1}^{J(p)} \int \chi_j(z) \Psi_q(z) e(\lambda_1 \Phi_q(z, p)) dz$$

and

$$R_6 = \int \left(1 - \sum_{j=1}^{J(p)} \chi_j(z)\right) \Psi_q(z) e(\lambda_1 \Phi_q(z, p)) dz.$$

It follows from integration by parts (Lemma A.2) and (B.16) that

$$(B.18) \quad R_6 \lesssim K^{-8d-4((5q+4)d+7-1/(d-1))N} \lambda_1^{-N}.$$

As to S_6 , for each $1 \leq j \leq J(p)$, let $\phi_j(z, p) = \Phi_q(z, p) - \Phi_q(Z_j, p)$. By Lemma A.4, if c_6 is sufficiently small then

$$(B.19) \quad \left| \int \chi_j(z) \Psi_q(z) e(\lambda_1 \Phi_q(z, p)) dz \right| = \left| \int \chi_j(z) \Psi_q(z) e(\lambda_1 \phi_j(z, p)) dz \right| \lesssim K^{-((5q+4)d+5-1/(d-1))/2} \lambda_1^{-d/2}.$$

Hence

$$(B.20) \quad S_6 \lesssim K^{-8d-((5q+4)d+5-1/(d-1))(d+1/2)} \lambda_1^{-d/2}.$$

Noticing that we have assumed $\lambda_1 \geq K^{-4(5q+4)d-29+4/(d-1)}$ in the Case 1, it is then easy to check that the bound (B.18) of R_6 is less than the bound (B.20) of S_6 if N is sufficiently large. Hence, by (B.12), (B.17), (B.20), and (B.18), we get the following bound of S_5

$$(B.21) \quad \begin{aligned} S_5 &\lesssim K^{8d} M_*^d ((K^{-8} \lambda_1 M_*^{-1})^d + 1) K^{-8d-((5q+4)d+5-1/(d-1))(d+1/2)} \lambda_1^{-d/2}, \\ &\lesssim K^{-((5q+4)d+5-1/(d-1))(d+1/2)} (K^{-8d} \lambda_1^{d/2} + M_*^d \lambda_1^{-d/2}). \end{aligned}$$

CASE 2: $\lambda_1 < K^{-4(5q+4)d-29+4/(d-1)}$.

Within this range of λ_1 , the assumption (B.5) implies $K^{-8} \lambda_1 M_*^{-1} < 1$, hence the trivial estimate of S_5 (together with (B.9) and (B.13)) yields

$$(B.22) \quad S_5 \lesssim M_*^d \leq K^{-(4(5q+4)d+29-4/(d-1))d/2} M_*^d \lambda_1^{-d/2}.$$

Combining the bounds of S_5 from Case 1 and 2 (namely, (B.21) and (B.22)) yields

$$S_5 \lesssim K^{-8d - ((5q+4)d+5-1/(d-1))(d+1/2)} \lambda_1^{d/2} + K^{-(4(5q+4)d+29-4/(d-1))d/2} M_*^d \lambda_1^{-d/2}.$$

Note that this bound of S_5 is larger than the bound (B.11) of R_5 no matter whether $\lambda_1 \leq 1$ or $\lambda_1 > 1$. It follows that

$$S_4 = S_5 + R_5 \lesssim K^{-(4q+13/2)d - ((5q+4)d+5-1/(d-1))(d+1/2)} (\mathcal{H} T M_*^{-q})^{d/2} + K^{-2d((5q+4)d+8-2q-1/(d-1))} (\mathcal{H} T M_*^{-q-2})^{-d/2},$$

where we have already used the definition of λ_1 .

Plugging this bound of S_4 into the Weyl-Van der Corput inequality that we used at the beginning gives

$$(B.23) \quad |S(T, M_*; G, F)|^Q \lesssim M_*^{dQ} H^{-1} + H^{(1-1/Q)d} T^{d/2} M_*^{d(Q-1-q/2)} K^{-(4q+13/2)d - ((5q+4)d+5-1/(d-1))(d+1/2)} + E,$$

where

$$E = K^{-2d((5q+4)d+8-2q-1/(d-1))} H^{-2+2/Q} T^{-d/2} M_*^{d(Q+q/2)}.$$

In order to balance the first two terms on the right side of (B.23) we let

$$H = c_5 (K^{\frac{2(5q+4)d^3 + (3q+19)d^2 - (13q+24)d - 6}{2(d-1)}} T^{-d/2} M_*^{(q/2+1)d})^{Q/(Q+(Q-1)d)}.$$

We then need to check that (B.8) is satisfied with this choice of H . First, $H > 1$ since we can assume

$$(B.24) \quad T < c_7 K^{(2(5q+4)d^3 + (3q+19)d^2 - (13q+24)d - 6)/(d(d-1))} M_*^{q+2}$$

with a sufficiently small c_7 (otherwise the trivial bound of $S(T, M_*; G, F)$, *i.e.* M_*^d , is better than (B.7)). Second, the assumption (B.6) implies the second inequality in (B.8).

With the choice of H as above and (B.24), we get

$$(B.25) \quad H^{-2+2/Q} T^{-d/2} M_*^{d(Q+q/2)} \leq K^{-\frac{2(5q+4)d^3 + (3q+19)d^2 - (13q+24)d - 6}{2(d-1)}} M_*^{(Q-1)d} H^{d-1}.$$

It then follows from (B.8) and (B.25) that

$$E \leq M_*^{Qd} H^{-1}.$$

Applying this bound to (B.23) finally yields the desired bound (B.7). \square

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